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Refined Cyclic Sieving

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Abstract. Reiner-Stanton-White (2004) defined the cyclic sieving phenomenon (CSP) associated to a finite cyclic group action and polynomial. A key example arises from the length generating function for minimal length coset representatives of a parabolic quotient of a finite Coxeter group. In type A, this result can be phrased in terms of the natural cyclic action on words of fixed content.

There is a natural notion of *refinement* for many CSP's. We formulate and prove a refinement of the aforementioned CSP arising from tracking the *cyclic descent type* of a word in addition to its content. The argument presented is completely different from Reiner-Stanton-White's representation-theoretic approach. It is combinatorial and largely, though not entirely, bijective.

A building block of our argument involves cyclic sieving for shifted subset sums, which also appeared in Reiner-Stanton-White. We give an alternate, largely bijective proof of a refinement of this result by extending some ideas of Wagon-Wilf (1994).

Keywords: cyclic sieving phenomenon, q-binomial coefficients, major index, words

1 Introduction

Since Reiner, Stanton, and White introduced the *cyclic sieving phenomenon* (CSP) in 2004 [4], the cyclic sieving phenomenon has become an important companion to any cyclic action on a finite set. Some remarkable examples of the CSP involve the action of a Springer regular element on Coxeter groups [4, Theorem 1.6], the action of Schutzenberger's promotion on Young tableaux of fixed rectangular shape [5], and the creation of new CSPs from old using multisets and plethysms with homogeneous symmetric functions [2, Proposition 8]. See [6] for a thorough introduction to the cyclic sieving phenomenon by Sagan.

Definition 1.1. Suppose C_n is a cyclic group of order n generated by σ_n, W is a finite set on which C_n acts, and $f(q) \in \mathbb{N}[q]$. We say the triple $(W, C_n, f(q))$ exhibits the *cyclic sieving phenomenon (CSP)* if for all $k \in \mathbb{Z}$,

$$\#W^{\sigma_n^k} := \#\{w \in W : \sigma_n^k \cdot w = w\} = f(\omega_n^k), \tag{1.1}$$

where ω_n is a fixed primitive n-th root of unity.

Our main result, Theorem 1.4, is a refinement of the type A case of [4, Theorem 1.6], which we now summarize; see Section 2 for missing definitions. Consider words in the alphabet $\mathbb{P} := \{1, 2, \ldots\}$. Given a word $w = w_1 \cdots w_n$, let $\mathrm{cont}(w)$ denote the *content* of w and write

$$W_\alpha := \{\text{words } w : \mathrm{cont}(w) = \alpha\} \tag{1.2}$$

for the set of words with content $\alpha \vDash n$. Write $\mathrm{maj}(w)$ for the *major index* of w. The cyclic group C_n acts on W_α by rotation.

In many instances of cyclic sieving, $f(q)$ is a statistic generating function on a set W. Given a statistic $\mathrm{stat}\colon W \to \mathbb{N}$, we form the generating function

$$W^{\mathrm{stat}}(q) := \sum_{w \in W} q^{\mathrm{stat}\, w} \in \mathbb{N}[q]. \tag{1.3}$$

We say two statistics $\mathrm{stat}, \mathrm{stat}' : W \to \mathbb{N}$ are *equidistributed* on W if $W^{\mathrm{stat}}(q) = W^{\mathrm{stat}'}(q)$. The following expresses an interesting result of Reiner, Stanton, and White in our notation.

Theorem 1.2. *[4, Proposition 4.4] Let $\alpha \vDash n$. The triple $(W_\alpha, C_n, W_\alpha^{\mathrm{maj}}(q))$ exhibits the CSP.*

Definition 1.3. A *refinement* of a CSP triple $(W, C_n, W^{\mathrm{stat}}(q))$ is a CSP triple $(V, C_n, V^{\mathrm{stat}}(q))$ where $V \subset W$ has the restricted C_n-action.

If $(V, C_n, V^{\mathrm{stat}}(q))$ refines $(W, C_n, W^{\mathrm{stat}}(q))$, then so does $(U, C_n, U^{\mathrm{stat}}(q))$, where $U = W - V$. Thus a refinement partitions W into smaller CSPs with the same statistic. In Section 3, we define a statistic on words, *flex*, which is universal in the sense that it refines to all C_n-orbits. Such universal statistics are essentially equivalent to the choice of total orderings for each orbit $\mathcal{O}$ of W.

We partition words of fixed content into fixed *circular descent type* (CDT). One computes $\mathrm{CDT}(w)$ by building up w through adding all 1's, 2's, …, and counting the number of *circular descents* introduced at each step. See Section 4 for the precise definition. We write the set of words with fixed content and circular descent type as

$$W_{\alpha,\delta} := \{\text{words } w : \mathrm{cont}(w) = \alpha, \mathrm{CDT}(w) = \delta\}. \tag{1.4}$$

Theorem 1.4. *Let $\alpha \vDash n$, and let δ be any weak composition. The triple*

$$(W_{\alpha,\delta}, C_n, W_{\alpha,\delta}^{\mathrm{maj}}(q))$$

exhibits the CSP.

Indeed, we derive an explicit product formula for $W_{\alpha,\delta}^{\mathrm{maj}}(q) \bmod (q^n - 1)$ involving q-binomial coefficients, see Theorem 4.11.

Reiner-Stanton-White's proof of Theorem 1.2 builds on a representation-theoretic result due to Springer [7, Proposition 4.5]. For Theorem 1.4, we have instead considered the problem of finding a bijective proof, which we interpret precisely as the problem of finding a bijective proof that flex and maj are equidistributed mod n on $W_{\alpha,\delta}$. Our argument is highly combinatorial, but it is not entirely bijective. Finding an explicit bijection would be quite interesting.

A key building block of our proof of Theorem 1.4 involves cyclic sieving on subsets. We describe a refinement of this result, Theorem 5.5, by restricting to subsets satisfying certain gcd requirements.

The rest of this extended abstract is organized as follows. In Section 2, we recall combinatorial background. In Section 3, we define the flex statistic. In Section 4, we decompose words with fixed content and circular descent type and summarize proofs of the product formula, Theorem 4.11, and our main result, Theorem 1.4. Section 5 gives an alternative proof of refined cyclic sieving on subsets using a method inspired by Wagon-Wilf [8]. Many of the proofs are omitted or summarized due to space constraints. They will be included in a forthcoming version of this article [1].

2 Combinatorial Background

In this section, we briefly introduce combinatorial notions on words and fix our notation. We use the alphabet of positive integers $\mathbb{P} := \{1, 2, \ldots\}$ throughout. A *word* w of length n is a sequence $w = w_1 w_2 \cdots w_n$ of *letters* $w_i \in \mathbb{P}$. Let $|w|$ denote the length of a word w. The *descent set* of w is $\mathrm{Des}(w) := \{1 \leq i < n : w_i > w_{i+1}\}$ and the number of descents is $\mathrm{des}(w) := \#\mathrm{Des}(w)$. The *major index* of w is $\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i$. The *inversion number* of w is $\mathrm{inv}(w) := \#\{1 \leq i < j \leq n : w_i > w_j\}$. The *cyclic descent set* of w is $\mathrm{CDes}(w) := \{1 \leq i \leq n : w_i > w_{i+1}\}$, where now the subscripts are taken mod n, and $\mathrm{cdes}(w) := \#\mathrm{CDes}(w)$. We use lower dots between letters to indicate cyclic descents and upper dots to indicate cyclic ascents throughout the paper, as in the following example.

Example 2.1. If $w = 155.3.155.3. = 1\dot{}5\dot{}531\dot{}5\dot{}53$, then $|w| = 8$, $\mathrm{Des}(w) = \{3, 4, 7\}$, $\mathrm{des}(w) = 3$, $\mathrm{CDes}(w) = \{3, 4, 7, 8\}$, $\mathrm{cdes}(w) = 4$, $\mathrm{maj}(w) = 14$, and $\mathrm{inv}(w) = 9$.

A *composition* or *weak composition* of n is a finite sequence $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ of non-negative integers adding up to n, typically denoted $\alpha \vDash n$. A *strong composition* additionally requires each $\alpha_i \neq 0$. The *content* of a word w, denoted $\mathrm{cont}(w)$, is the sequence α whose j-th part is the number of j's in w. For a word w of length n, $\mathrm{cont}(w)$ is a weak composition of n. We write

$$W_n := \{\text{words } w : |w| = n\}, \tag{2.1}$$

$$W_\alpha := \{w \in W_n : \mathrm{cont}(w) = \alpha\}. \tag{2.2}$$

The cyclic group $C_n := \langle \sigma_n \rangle$ of order n acts on W_n and W_α by

$$\sigma_n \cdot w_1 \cdots w_{n-1} w_n := w_n w_1 \cdots w_{n-1}.$$

The set of all words in $\mathbb{P}$ is a monoid under concatenation. A word is *primitive* if it is not a power of a smaller word. Any non-empty word w may be written uniquely as $w = v^f$ for $f \geq 1$ with v primitive. We call $|v|$ the *period* of w, written $\operatorname{period}(w)$, and f the *frequency* of w, written $\operatorname{freq}(w)$. The orbit of $w \in W_n$ under rotation is a *necklace*, usually denoted $[w]$. We have $\operatorname{period}(w) = \#[w]$ and $\operatorname{freq}(w) \cdot \operatorname{period}(w) = |w|$. Content, primitivity, period, frequency, and cdes are all constant on necklaces.

Example 2.2. The necklace of $w = 15531553 = (1553)^2$ is

$$[w] := \{15531553, 55315531, 53155315, 31553155\} \subset W_{(2,0,2,0,4)} \subset W_8,$$

which has period 4, frequency 2, and cdes 4.

Reiner-Stanton-White gave several equivalent conditions for a triple $(W, C_n, f(q))$ to exhibit the CSP. In place of (1.1) in Definition 1.1, we may instead require

$$f(q) \equiv \sum_{\text{orbits } \mathcal{O} \subset W} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1} \pmod{q^n - 1}, \tag{2.3}$$

where the sum is over all orbits $\mathcal{O}$ under the action of C_n on W, and for $d \mid n$,

$$\frac{q^n - 1}{q^d - 1} = \sum_{i=0}^{n/d-1} q^{id} \not\equiv 0 \pmod{q^n - 1}. \tag{2.4}$$

We refer the interested reader to [4, Proposition 2.1] for the proof of the equivalence of (1.1) and (2.3).

For a set S, write

$$\binom{S}{k} := \{k\text{-element subsets of } S\}, \quad \left(\!\!\binom{S}{k}\!\!\right) := \{k\text{-element multisubsets of } S\}.$$

We use the following q-analogues for $n, k \in \mathbb{Z}_{\geq 0}$, where "sum” is the sum of the elements in a set or multiset:

$$[n]_q = 1 + q + \cdots + q^{n-1}, \tag{2.5}$$

$$[n]_q! = [n]_q[n-1]_q \dots [1]_q, \tag{2.6}$$

$$\binom{n}{\alpha_1, \dots, \alpha_m}_q = \frac{[n]_q!}{[\alpha_1]_q! \dots [\alpha_m]_q!}, \tag{2.7}$$

$$\binom{n}{k}_q = \binom{n}{k, n-k}_q = q^{-\binom{k}{2}} \binom{[0, n-1]}{k}^{\mathrm{sum}}(q), \tag{2.8}$$

$$\left(\!\!\binom{n}{k}\!\!\right)_q = \binom{n+k-1}{k}_q = \left(\!\!\binom{[0, n-1]}{k}\!\!\right)^{\mathrm{sum}}(q). \tag{2.9}$$

For $\alpha = (\alpha_1, \dots, \alpha_m) \vDash n$, maj and inv are equidistributed on W_α with

$$W_\alpha^{\mathrm{maj}}(q) = W_\alpha^{\mathrm{inv}}(q) = \binom{n}{\alpha_1, \dots, \alpha_m}_q \tag{2.10}$$

by a classical result of MacMahon [3]. Despite (2.10), maj and inv are not equidistributed modulo n on $W_{\alpha,\delta}$ in general, so $(W_{\alpha,\delta}, C_n, W_{\alpha,\delta}^{\mathrm{inv}}(q))$ does not exhibit the CSP.

3 The Flex Statistic

Definition 3.1. Let $\mathrm{lex}(w)$ denote the index at which w appears when lexicographically ordering the necklace $[w]$, starting from 0. Let *flex* be the product

$$\mathrm{flex}(w) := \mathrm{freq}(w)\,\mathrm{lex}(w).$$

For example, the necklace in Example 2.2 has lex statistics $0, 3, 2, 1$, respectively, so that $\mathrm{lex}(55315531) = 3$ and $\mathrm{flex}(55315531) = 2 \cdot 3 = 6$. The following is a direct consequence of (2.3).

Lemma 3.2. *For any necklace N of length n, $(N, C_n, N^{\mathrm{flex}}(q))$ exhibits the CSP.*

Thus, for any set $W \subset W_n$ on which C_n acts, $(W, C_n, W^{\mathrm{flex}}(q))$ exhibits the CSP. In this sense, flex is a "universal CSP statistic on words". By contrast, for instance when $N = [123123]$, the triple $(N, C_6, N^{\mathrm{maj}}(q))$ does not exhibit the CSP. In this sense, maj is a not a universal CSP statistic on words. Indeed, it follows from Lemma 3.2 that Theorem 1.4 is equivalent to maj and flex being equidistributed modulo n on $W_{\alpha,\delta}$. It would be interesting to find a natural, explicit bijection proving this version of Theorem 1.4.

Open Problem 3.3. *For $\alpha \vDash n$ and any composition δ, find a bijection $\varphi : W_{\alpha,\delta} \to W_{\alpha,\delta}$ satisfying*

$$\operatorname{maj}(\varphi(w)) \equiv \operatorname{flex}(w) \pmod{n}. \tag{3.1}$$

4 Cyclic Descent Type and Major Index

The main purpose of this section is to give a product formula for $W^{\operatorname{maj}}_{\alpha,\delta}(q)$ modulo $q^n - 1$, Theorem 4.11. The $q = 1$ specialization gives a formula for $\#W_{\alpha,\delta}$, Lemma 4.10. We conclude this section by summarizing our proof of the main result, Theorem 1.4, given Theorem 4.11.

4.1 Cyclic Descent Type

We begin by defining cyclic descent type. Throughout this section, w is a word with $\operatorname{cont}(w) = \alpha = (\alpha_1, \ldots, \alpha_m)$, a strong composition of $n \geq 1$, and k circular descents, so that $m = \max(w)$. Let $w^{(i)}$ denote the subsequence of w with all elements larger than i removed. We have a "filtration"

$$\varnothing \preceq 1^{\alpha_1} = w^{(1)} \preceq w^{(2)} \preceq \cdots \preceq w^{(m-1)} \preceq w^{(m)} = w,$$

where $u \preceq v$ means that u is a subsequence of v. The *cyclic descent type* of a word w, denoted $\operatorname{CDT}(w)$, is the sequence which tracks the number of new cyclic descents at each stage of the filtration:

$$\operatorname{CDT}(w) := (\operatorname{cdes} w^{(1)}, \operatorname{cdes} w^{(2)} - \operatorname{cdes} w^{(1)}, \ldots, \operatorname{cdes} w^{(m)} - \operatorname{cdes} w^{(m-1)}). \tag{4.1}$$

Note CDT is constant on necklaces since rotating w rotates each $w^{(i)}$. Also, $\operatorname{CDT}(w)$ is a weak composition of $\operatorname{cdes}(w) = k$. Recall our earlier notation (1.4):

$$W_{\alpha,\delta} := \{w \in W_n : \operatorname{cont}(w) = \alpha, \operatorname{CDT}(w) = \delta\}.$$

Example 4.1. Suppose $w = 143124114223$, so $\operatorname{cont}(w) = (4, 3, 2, 3)$,

$$\begin{aligned}
w^{(1)} &= 1111 & \operatorname{cdes} w^{(1)} &= 0,\\
w^{(2)} &= 112.1122. & \operatorname{cdes} w^{(2)} &= 2,\\
w^{(3)} &= 13.12.11223. & \operatorname{cdes} w^{(3)} &= 3,\\
w^{(4)} &= 14.3.124.114.223. & \operatorname{cdes} w^{(4)} &= 5.
\end{aligned}$$

Hence, $\operatorname{CDT}(143124114223) = (0, 2 - 0, 3 - 2, 5 - 3) = (0, 2, 1, 2)$.

4.2 Runs, Falls, and Major Index

We describe a way to create words with a fixed content and CDT in terms of insertions into *runs* and *falls*. We organize this procedure into a tree whose edges are labelled by sets and multisets (Definition 4.8). Lemma 4.7 describes changes in the major index upon traversing an edge of this tree, which gives Theorem 4.11 when iteratively applied.

Definition 4.2. A *fall* in w is a maximal set of distinct indexes $i, i+1, \ldots, j-1, j$ such that $w_i > w_{i+1} > \cdots > w_j$, where we take indices modulo n. A *run* in a non-constant word w is a maximal set of distinct indexes $i, i+1, \ldots, j$ such that $w_i \leq w_{i+1} \leq \cdots \leq w_j$, where we take indices modulo n. The constant word $w = \ell^n$ by convention has no runs, though this case will not actually appear in our arguments.

Each letter is part of a unique run and a unique fall, except when $w = \ell^n$ is constant. It is easy to see that with these conventions, w has $n - \operatorname{cdes}(w)$ falls and $\operatorname{cdes}(w)$ runs. We index falls from 0 starting at the fall containing the first letter of w, and we do the same with runs.

Example 4.3. Let $w = 26534611 = 26.5.346.11 = 2^{\cdot}653^{\cdot}4^{\cdot}61^{\cdot}1^{\cdot}$, where lower dots indicate cyclic descents and upper dots indicate cyclic weak ascents. The three runs of w are $1126, 5, 346$, with respective indexes $0, 1, 2$. The five falls of w are $2, 653, 4, 61, 1$ with respective indexes $0, 1, 2, 3, 4$.

Definition 4.4. Fix a letter ℓ not in w. Pick a subset F of the falls of w. We *insert ℓ into the falls F* by successively inserting ℓ into each fall $w_i > w_{i+1} > \cdots > w_j$ in F so that $w_i \cdots \ell \cdots w_j$ is still decreasing.

Similarly, we may fix a letter ℓ and pick a *multisubset* R of the runs of w. We *insert ℓ into the runs R* by successively inserting ℓ into each run $w_i \leq w_{i+1} \leq \cdots \leq w_j$ in R so that $w_i \cdots \ell \cdots w_j$ is still weakly increasing.

In either case, if we have a choice between inserting a letter at either the beginning of w or at the end of w, we choose to insert at the beginning of w. When inserting ℓ into a run already containing ℓ, the resulting word is independent of precisely which of the possible positions is used.

Example 4.5. Let $w = 2^{\cdot}653^{\cdot}4^{\cdot}61^{\cdot}1^{\cdot}$. Insert 7 into the falls of w with indexes 0 and 3 to successively obtain $\underline{7}2^{\cdot}653^{\cdot}4^{\cdot}61^{\cdot}1^{\cdot}$ and then $w' := 72^{\cdot}653^{\cdot}4^{\cdot}\underline{7}61^{\cdot}1^{\cdot}$. Note that $w' = 7.26.5.347.6.11$ has two more runs than w. Now insert 7 into the runs of w' with multiset of indexes $\{0, 2, 3, 3\}$ to successively obtain $\underline{7}7.26.5.347.6.11$, $77.26.5\underline{7}.347.6.11$, $77.26.57.347.6.11$, and $w'' := 77.26.57.34\underline{7}7.6.11$.

It will shortly prove convenient to restrict to words ending in a 1. Let

$$\widetilde{W}_n = \{w \in W_n : w \text{ ends in a } 1\},$$
$$\widetilde{W}_{\alpha,\delta} = \{w \in W_{\alpha,\delta} : w \text{ ends in a } 1\}.$$

Inserting a letter into a subset of the falls and then a multisubset of the runs forms the basis of our procedure for constructing words in $\widetilde{W}_{\alpha,\delta}$.

Definition 4.6. Fix $w \in \widetilde{W}_n$, a letter ℓ not in w, and

$$F \subset [0, |w| - \operatorname{cdes} w - 1] \qquad \text{and} \qquad R \underset{\text{mult.}}{\subset} [0, \operatorname{cdes} w + |F| - 1]$$

where $\underset{\text{mult.}}{\subset}$ denotes a multisubset. Let w' be obtained by inserting ℓ into falls F of w, and let w'' be obtained by inserting ℓ into runs R of w'. We say w'' is obtained by *inserting the triple* (ℓ, F, R) into w. In particular, $\operatorname{cdes} w'' = \operatorname{cdes} w + |F|$. By keeping track of the effect of each insertion on maj, one can show the following.

Lemma 4.7. *Suppose w'' is obtained by inserting the triple (ℓ, F, R) into $w \in \widetilde{W}_n$. Then,*

$$\operatorname{maj}(w'') - \operatorname{maj}(w) = \binom{|F|+1}{2} + (\operatorname{cdes} w)(|F| + |R|) + |F||R| + \sum_{f \in F} f - \sum_{r \in R} r. \tag{4.2}$$

Definition 4.8. Fix a strong composition α of n and $\delta \vDash k$, and let $n_\ell := \alpha_1 + \cdots + \alpha_\ell$ and $k_\ell := \delta_1 + \cdots + \delta_\ell$. Construct a rooted, vertex- and edge-labeled tree $T_{\alpha,\delta}$ recursively as follows. Begin with root 1^{α_1}. For $\ell = 2, 3, \ldots$, for each leaf w, and for each triple (ℓ, F, R) with

$$F \in \binom{[0, n_{\ell-1} - k_{\ell-1} - 1]}{\delta_\ell} \qquad \text{and} \qquad R \in \left(\!\!\binom{[0, k_\ell - 1]}{\alpha_\ell - \delta_\ell}\!\!\right),$$

add an edge labeled by (ℓ, F, R) from w to w'' where w'' is obtained by inserting the triple (ℓ, F, R) into w.

Example 4.9. Let $\alpha = (4, 2, 3)$ and $\delta = (0, 2, 1)$. The following is the subgraph of $T_{\alpha,\delta}$ consisting of paths from the root to leaves that are rotations of 112113323:

$$\begin{tikzcd}
 & & & 1111 \arrow[dll, "{(2,\{0,2\},\varnothing)}"'] \arrow[drr, "{(2,\{1,3\},\varnothing)}"] & & & \\
 & 211211 \arrow[dl, "{(3,\{0\},\{0,1\})}"'] \arrow[dr, "{(3,\{2\},\{1,2\})}"] & & & & 121121 \arrow[dl, "{(3,\{1\},\{0,1\})}"'] \arrow[dr, "{(3,\{3\},\{1,2\})}"] & \\
332311211 & & 211332311 & & 133231121 & & 121133231
\end{tikzcd}$$

For this full $T_{\alpha,\delta}$, the root has $\binom{4}{2} = 6$ children since 1111 has 4 falls. Each child of the root itself has $\binom{4}{1}\left(\!\binom{2}{2}\!\right) = 12$ children. Hence, $T_{\alpha,\delta}$ has 72 leaves. The cyclic rotations of 112113323 appearing as leaves in Example 4.9 are exactly those ending in 1.

Lemma 4.10. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a strong composition of n and $\delta = (0, \delta_2, \ldots, \delta_m) \vDash k$. Then*

$$\#W_{\alpha,\delta} = \frac{n}{\alpha_1} \prod_{i=2}^{m} \binom{n_{i-1} - k_{i-1}}{\delta_i} \left(\!\!\binom{k_i}{\alpha_i - \delta_i}\!\!\right) \tag{4.3}$$

In particular, $W_{\alpha,\delta} \neq \varnothing$ if and only if

$$\begin{aligned} 0 \leq \delta_i &\leq \alpha_i && \text{for } 1 \leq i \leq m, \\ \delta_1 + \cdots + \delta_{i+1} &\leq \alpha_1 + \cdots + \alpha_i && \text{for } 1 \leq i < m. \end{aligned}$$

Theorem 4.11. *Suppose $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a strong composition of n and $\delta = (\delta_1, \ldots, \delta_m) \vDash k$. Set $n_i := \alpha_1 + \cdots + \alpha_i$, $k_i := \delta_1 + \cdots + \delta_i$, and $d = \gcd(n, k)$. Then, modulo $q^n - 1$,*

$$W^{\mathrm{maj}}_{\alpha,\delta}(q) \equiv \frac{d}{\alpha_1} \left(\frac{q^n - 1}{q^d - 1} \right) q^{\binom{k}{2} + \sum_{i=2}^{m} \binom{\delta_i}{2} - \alpha_1} \prod_{i=2}^{m} \binom{n_{i-1} - k_{i-1}}{\delta_i}_q \left(\!\!\binom{k_i}{\alpha_i - \delta_i}\!\!\right)_q \tag{4.4}$$

Lemma 4.10 and Theorem 4.11 are consequences of the structure of the tree $T_{\alpha,\delta}$ and Lemma 4.7.

4.3 Proof Sketch of the Main Theorem

From the explicit formula in Theorem 4.11, one may show the coefficient of q^i in $W^{\mathrm{maj}}_{\alpha,\delta}(q) \pmod{q^n - 1}$ depends only on i modulo $g := \gcd(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_m)$. We then show the triple $(W_{\alpha,\delta}, \langle \sigma_n^{n/g} \rangle, W^{\mathrm{maj}}_{\alpha,\delta}(q))$ exhibits the CSP using products of CSP's on sets and multisets. Finally, we combine these two observations to extend this CSP to C_n to achieve Theorem 1.4.

5 A Refinement of the Cyclic Sieving Phenomenon on Subsets

Our proof of Theorem 1.4 uses products involving the following CSP triples, first observed by Reiner, Stanton, and White.

Theorem 5.1. *[4, Theorem 1.1]. Let C_n act on $\binom{[0,n-1]}{k}$ by incrementing values modulo n. Then*

$$\left(\binom{[0, n-1]}{k}, C_n, \binom{n}{k}_q \right)$$

exhibits the CSP.

We refer the reader to Sections 3 and 4 of [4] for proofs of Theorem 5.1 via representation theory or direct calculation, respectively. We give an alternative proof inspired by a method of Wagon and Wilf [8, §3], who characterized when the subset sum statistic is equidistributed modulo m. The main novelty of our approach is Theorem 5.5, which gives a refinement of a related CSP using a shifted sum statistic to subsets satisfying certain gcd requirements. Throughout this section, we fix n and k and let $S := \binom{[0,n-1]}{k}$.

Definition 5.2. For all $r \mid n$ and $j \in [1, \frac{n}{r}]$, let

$$I_r^j := [(j-1)r, jr-1],$$

which we call an *r-interval*. Let C_r, the cyclic group of order r, act on $S = \binom{[0,n-1]}{k}$ by simultaneous rotation of r-intervals. For example, when $n = k = 6$, the generator for C_3 acts on the 6-interval by the permutation $(012)(345)$. Finally, let

$$\mathrm{sum}'(A) := \sum_{a \in A} a - \sum_{i=0}^{k-1} i = \sum_{a \in A} a - \binom{k}{2}. \tag{5.1}$$

Recall from (2.8) that

$$\binom{[0,n-1]}{k}^{\mathrm{sum}'}(q) = \binom{n}{k}_q. \tag{5.2}$$

Using (5.2), we may restate Theorem 5.1 as saying that $(S, C_n, S^{\mathrm{sum}'}(q))$ exhibits the CSP, where C_n acts on S by rotation. Let C_r' denote the unique subgroup of size r in C_n. The action of C_r on S in Definition 5.2 differs from this restricted action of C_r', but they are easily seen to be isomorphic. Thus, $(S, C_r', S^{\mathrm{sum}'}(q))$ exhibits the CSP as well. We will thus use the action of C_r on S in Definition 5.2 for the rest of this section.

Definition 5.3. Let

$$G_{a,b} = \{A \in S : \gcd(a, \#(A \cap I_a^1), \#(A \cap I_a^2), \ldots, \#(A \cap I_a^{n/a})) = b\}.$$

In particular, $G_{a,b}$ is empty unless $b \mid a$. Also, for any divisibility chain $D = d_p \mid d_{p-1} \mid \cdots \mid d_1 \mid n$, let

$$G_D = G_{n,d_1} \cap G_{d_1,d_2} \cap \cdots \cap G_{d_{p-1},d_p}. \tag{5.3}$$

Example 5.4. If $n = 4, k = 2$, then

$$G_{1|2|4} = G_{4,2} \cap G_{2,1} = \{\{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}\},$$
$$G_{2|2|4} = G_{4,2} \cap G_{2,2} = \{\{0,1\}, \{2,3\}\}.$$

Theorem 5.5. *Fix $n, k \in \mathbb{Z}_{\geq 0}$, let $S := \binom{[0,n-1]}{k}$, and let D be a divisibility chain beginning with $e \mid d$ and ending with n. Then, C_d acts on S and G_D by simultaneous rotation of d-intervals, and*

$$(G_D, C_d, G_D^{\mathrm{sum}'}(q))$$

exhibits the CSP and refines the CSP triple $(S, C_d, S^{\mathrm{sum}'}(q))$.

Example 5.6. Suppose $n = 4, k = 2, D = 1 \mid 2 \mid 4$, so $G_D = G_{4,2} \cap G_{2,1}$. Then, G_D has C_2 orbits $\{\{0,2\},\{1,3\}\}$ and $\{\{1,2\},\{0,3\}\}$ and

$$G_D^{\mathrm{sum}'}(q) = q^1 + 2q^2 + q^3 \equiv 2 \cdot \frac{q^2-1}{q-1} \pmod{q^2-1}.$$

Thus, $(G_{1|2|4}, C_2, G_{1|2|4}^{\mathrm{sum}'}(q))$ exhibits the CSP.

Theorem 5.5 includes Theorem 5.1 in the special case when $D = \gcd(n,k) \mid n$. Our proof of Theorem 5.5 proceeds by induction on d. The cases where $d = 1$ or $e = d$ are easy to verify on their own. Otherwise, we have $e < d$, and we use the induction hypothesis to deduce $(G_D, C_e, G_D^{\mathrm{sum}'}(q))$ exhibits the CSP.

In order to extend this CSP up to C_d, we replace C_e by C_e', the unique subgroup of size e in C_d, which is valid since the actions of C_e and C_e' on G_D are isomorphic. We then apply a straightforward extension lemma to prove Theorem 5.5.

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